Reply to a paper by M T Barlow and S J Taylor (Fractional dimension of sets in discrete systems)

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COMMENT

Reply to a paper by M T Barlow and S J Taylor

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Abstract. It is pointed out that the different definitions of the fractional dimension of an arbitrary subset A of Z^d, given by Naudts and by Barlow and Taylor, are interrelated but obtained using completely different criteria.

The different definitions of fractional dimension of an arbitrary subset A of Z^d that are used in [1, 2] are interrelated. The mass dimension, the dimension introduced in [1], and the dimension introduced by Barlow and Taylor in [2] are denoted d_M, d_N, and d_H respectively.

Both d_N and d_H are based on forming optimal covers of finite parts of A. Given a finite subset S of Z^d and positive numbers b and α let

$$m_\alpha(A, S, b) = \min \sum_{i=1}^{m} s(B_i)^\alpha$$

where the B_i are cubes of size $s(B_i) \geq b$, and where the minimum is taken over all covers $B_1, \ldots, B_m$ of $A \cap S$ with m arbitrary. The quantity $m_\alpha(A, S, b)$ is an estimate for the α-dimensional volume of the set $A \cap S$.

The criteria used to define d_N and d_H are completely different.

Criterion for d_N: The density of points in $A \cap S$ is $|A \cap S| / m_\alpha(A, S, b)$; if $\alpha < d_N$ then the density diverges in the limit $S \to \infty$, for all values of the coarse-graining parameter b.

Criterion for d_H: The α-dimensional volumes of $A \cap S$ and S are compared; if $d_H < \alpha < d$ then the ratio $m_\alpha(A, S, 1)/m_\alpha(S, S, 1)$ tends to zero in the limit $S \to \infty$.

Let us now assume that the mass dimension d_M of the set A exists. Then $|A \cap S|$ behaves as $|S|^{d_M/d}$ for large S. Using the relation $m_\alpha(S, S, 1) = |S|^{\alpha/d}$ one obtains

$$\frac{|A \cap S|}{m_\alpha(A, S, 1)} \times \frac{m_\alpha(A, S, 1)}{m_\alpha(S, S, 1)} \sim |S|^{(d_M - \alpha)/d}.$$

The two factors of the left-hand side are precisely the relevant quantities in the definitions of d_N and d_H. In 'nice' scaling situations one has $d_M = d_N = d_H$. However, if d_H is strictly smaller than d_M, as is the case for one of the examples in [2], then by the above relation one expects d_N to be strictly larger than d_M.

Both dimensions d_M and d_N are based on expressions for the density of points in A. Since densities are not additive it is not a surprise that d_M and d_N are not monotonous.

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Finally, let me stress the importance in practical situations of the coarse-graining parameter b. Experimentally only a finite part $A \cap S$ of any self-similar set A is known—usually S is a cube of rather limited size. Two techniques are presented in [1] for estimating the dimension of A from knowledge of $m_\alpha(A, S, b)$ for fixed S as a function of α and b. They are based on the following observations:

(i) $m_\alpha(A, S, b)$ is expected to be independent of b whenever $\alpha < d_H$ (see the numerical example in [1])

(ii) for values of α slightly above d_M one expects scaling behaviour: $m_\alpha(A, S, b) \sim b^\alpha$ with $\alpha = d - d_M$ (an example is worked out in [3]).

References

[1] Naudts J 1988 *J. Phys. A: Math. Gen.* **21** 447

[2] Barlow M T and Taylor S J 1989 *J. Phys. A: Math. Gen.* **22**

[3] Naudts J 1987 *Time-dependent Effects in Disordered Materials* ed R Pynn and T Riste (New York: Plenum) p 339